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Line bundles over families of (super) Riemann surfaces. I: The non-graded case *

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A first step towards a systematic theory of relative line bundles over SUSY-curves is presented. In this paper we deal with the case of relative line bundles over families of ordinary Riemann surfaces. Generalizations of the Gauss–Bonnet theorem and of the flatness theorem for line bundles are discussed.

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1. Introduction

Polyakov's approach to the computation of quantum scattering amplitudes in the bosonic string theory admits a nice geometric interpretation in terms of integrals over the moduli space of Riemann surfaces [1]. One expects that superstrings (strings with bose and fermi degrees of freedom) can be dealt with in a similar way by introducing a suitable $\mathbb{Z}_2$-graded analog of a Riemann surface.

Physical arguments [5] suggest that the specification of a "super Riemann surface" should include:

(i) an ordinary Riemann surface X;
(ii) a spin structure on X;
(iii) a set of gravitino fields on X.

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A convenient geometric setting for dealing with structures of this kind is provided by the Berezin–Leĭtes–Kostant graded manifolds (or supermanifolds). If one considers a (1,1) dimensional complex analytic graded manifold $(X, \mathcal{B}_X)$, then $(X, (\mathcal{B}_X)_0)$ is just an ordinary Riemann surface; $(\mathcal{B}_X)_1$ is in general a line bundle over X, and one can require that it is a spin structure on X. In order to accommodate the last datum, i.e. a set of gravitino fields, one can in a sense consider *a family of super Riemann surfaces*, also called *SUSY-curve* [12,13].

The study of the geometry of these objects involves the extension of several constructions which are encountered in the usual theory of Riemann surfaces. We center our attention on some facts concerning (relative) line bundles over SUSY-curves; in particular, we deal with the extension of two results: the Gauss–Bonnet theorem, and the fact that a holomorphic line bundle on a Riemann surface is flat if and only if its Chern class vanishes. This is done in two steps: the first, which is dealt with in the present paper, is the generalization of these results to the case of a family of ordinary Riemann surfaces; the second deals with the graded case, and forms the object of the following paper [3].

It should be noticed that some material we present here can be found in some classical works by Mumford and others; however, the techniques used there are quite different from those we employ here, which reduce to some complex geometry and basic homological algebra. This makes the extension to the graded setting quite straightforward.

Let us describe briefly the contents of this paper. After a cursory presentation of the basic definitions, we develop in section 2 the notions of relative de Rham theory and of fiberwise integration over a family of smooth manifolds; a relative Serre duality for families of complex analytic manifolds is also stated. In section 3 we introduce, basically following Grothendieck [7,8], the notions of Picard sheaf and relative Picard group of a family of complex manifolds; also the concept of Chern class finds a natural generalization to the relative context. A relative flatness property is stated, and it is proved that a section of the Picard sheaf is flat if and only if its relative Chern class vanishes. Moreover, we show that the relative Chern class can be represented in terms of the fiberwise integral of a curvature form.

The main results of this paper and of the following one were announced without proofs in ref. [2]. The graded version of some of them already appeared in ref. [6]; however, the transition from the absolute to the relative case is not obtained—as there claimed—simply by replacing the sheaf cohomology functor with the higher direct image functor. Indeed, the category of families of graded manifolds is essentially different from that of single graded manifolds [14,17]. Also, the definition of relative flatness as given in ref. [6] seems to be inadequate, and does not allow for a satisfatory discussion of the validity of the flatness theorem for SUSY-curves.

2. Families of smooth and complex analytic manifolds

2.1. PRELIMINARIES

We shall denote by $(X, \mathcal{O}_X)$ a complex analytic manifold, and by $(X, \mathcal{C}_X)$ a real differentiable manifold, where $\mathcal{C}_X$ is the sheaf of $\mathbb{C}$-valued smooth functions on X. All manifolds will be assumed to be connected. Every n-dimensional complex manifold determines an underlying $2n$-dimensional real manifold; if $(z_1, \ldots, z_n)$ are local coordinates in $(X, \mathcal{O}_X)$, then $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ are local coordinates for $(X, \mathcal{C}_X)$. We shall use the notation $(X, \mathcal{R}_X)$ interchangeably for a real or complex manifold, denoting in either case the local coordinate systems by $\{z_i\}$. The sheaf $\mathcal{D}er\, X$ is the sheaf of derivations of $\mathcal{R}_X$, and Ω^1_X is the sheaf of differentials of $\mathcal{R}_X$, i.e. the $\mathcal{R}_X$-dual module of $\mathcal{D}er\, X$; these are locally free, and any local coordinate system $\{z_i\}$ induces local bases $\{\partial/\partial z_i\}$ of $\mathcal{D}er\, X$ and $\{dz_i\}$ of Ω^1_X. The corresponding vector bundles are the tangent bundle TX and the cotangent bundle T^*X, respectively. From the inverse function theorem one obtains that a set $\{z_i\}$ of sections of $\mathcal{R}_X$ is a local coordinate system around $x \in X$ if and only if the differentials of those sections at x, i.e. the elements $\{d_x z_i\}$, are a basis for $T^*_x X$.

A manifold morphism $\pi : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is said to be a *submersion* if for all $x \in X$ the induced morphism $\pi^*_x : T^*_{\pi(x)} Y \to T^*_x X$ is injective, or, equivalently, if for all $x \in X$ there are open neighborhoods U of x and V of $\pi(x)$, with $\pi(U) \subset V$, such that (1) the natural map $\pi^* : \mathcal{R}_Y(V) \to \mathcal{R}_X(U)$ is injective; and (2) every coordinate system $\{w_i\}$ in V can be completed to a coordinate system $\{w_i, z_j\}$ in U.

The morphism $\pi : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ is said to be *proper* if it is such in the sense of topological spaces; *flat* if $\pi_* \mathcal{R}_X$ is a flat sheaf of $\mathcal{R}_Y$-modules. In the complex analytic case, if $\pi_* \mathcal{O}_X$ is coherent (which for instance happens when π is proper via Grauert's cohomology base change theorem) the flatness of π implies that $\pi_* \mathcal{O}_X$ is locally free over $\mathcal{O}_Y$.

For all $y \in Y$, the fiber $X_y = \pi^{-1}(y)$ of a proper morphism π is a compact space, which is endowed with a manifold structure given by the structure sheaf

$$\mathcal{R}_{X_y} = (\mathcal{R}_X / \hat{\mathfrak{m}}_y)|_{X_y} \simeq \mathcal{R}_X|_{X_y} \otimes_{(\mathcal{R}_Y)_y} k(y),$$

where $\hat{\mathfrak{m}}_y$ is the ideal of $\mathcal{R}_X$ generated by the inverse image of $\mathfrak{m}_y$; here $\mathfrak{m}_y$ denotes the maximal ideal of $(\mathcal{R}_Y)_y$, and also the naturally associated sheaf of ideals of $\mathcal{R}_Y$. Moreover, $k(y)$ is the residual field at y, i.e., $k(y) = (\mathcal{R}_Y)_y / \mathfrak{m}_y \simeq \mathbb{C}$. The manifold $(X_y, \mathcal{R}_{X_y})$ can also be regarded as the fibered product $X \times_Y \{y\}$.

We say that a morphism $\pi : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ has *universally connected fibers* if $\pi_* \mathcal{R}_X \simeq \mathcal{R}_Y$; this implies indeed that all fibers X_y are connected, and that this property is preserved under any change of the basis $(Y, \mathcal{R}_Y)$.

A *family of manifolds* is a proper submersion $\pi : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$. In the complex analytic case we also assume that π is flat and has universally connected fibers. The *relative dimension* of the family is the number $\dim X - \dim Y$, which corresponds to the dimension of each fiber. A complex analytic family of relative dimension 1 is called a *family of Riemann surfaces*. A *relative coordinate system* is a local coordinate system $\{w_i, z_j\}$ on X such that $\{w_i\}$ is a local coordinate system on Y.

If $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a family of complex manifolds of relative dimension n there is a vanishing theorem $R^k\pi_*\mathcal{O}_X = 0$ for all $k > n$, where $R^k\pi_*$ denotes the kth higher direct image functor associated with π_*.

A *morphism of families* between $\pi' : (X', \mathcal{R}_{X'}) \to (Y', \mathcal{R}_{Y'})$ and $\pi : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$—which we shall briefly denote by $f : (X'/Y') \to (X/Y)$—is a pair of morphisms $f : (X', \mathcal{R}_{X'}) \to (X, \mathcal{R}_X)$ and $f' : (Y', \mathcal{R}_{Y'}) \to (Y, \mathcal{R}_Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} (X', \mathcal{R}_{X'}) & \xrightarrow{f} & (X, \mathcal{R}_X) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ (Y', \mathcal{R}_{Y'}) & \xrightarrow[f']{} & (Y, \mathcal{R}_Y) \end{array} .$$

The *sheaf of relative derivations* of a family $\pi : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$, denoted $\mathcal{D}er\,(X/Y)$, is the subsheaf of $\mathcal{D}er\,X$ whose elements vanish on $\mathcal{R}_Y$, that is, the following exact sequence holds:

$$0 \longrightarrow \mathcal{D}er\,(X/Y) \longrightarrow \mathcal{D}er\,X \longrightarrow \pi^*\mathcal{D}er\,Y \longrightarrow 0 .$$

The dual module of $\mathcal{D}er\,(X/Y)$, denoted $\Omega^1_{X/Y}$, is the *sheaf of relative differentials* of the family, and one has an exact sequence

$$0 \longrightarrow \pi^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 .$$

Any relative coordinate system $\{w_i, z_j\}$ induces local bases $\{\partial/\partial z_j\}$ of $\mathcal{D}er\,(X/Y)$ and $\{dz_j\}$ of $\Omega^1_{X/Y}$.

For every fiber $X_y = \pi^{-1}(y)$ there are identifications

$$\Omega^1_{X_y} \simeq (\Omega^1_{X/Y}/\hat{\mathfrak{m}}_y\Omega^1_{X/Y})|_{X_y} \simeq \Omega^1_{X/Y}|_{X_y} \otimes_{(\mathcal{R}_Y)_y} k(y) ,$$

namely, the differentials of the fibers can be obtained from the relative differentials by restricting to the fibers and taking values; in this sense, if $\{w_i, z_j\}$ is a relative coordinate system on the family, the $\{dz_j\}$'s are a local basis for $\Omega^1_{X_y}$, and $\{z_j\}$ is a local coordinate system for the fiber.

A family $\pi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ of real manifolds is said to be *orientable* if the relative cotangent bundle—namely, the vector bundle associated with the locally free module $\Omega^1_{X/Y}$—is orientable, which amounts to saying that the highest exterior power of $\Omega^1_{X/Y}$ has a global nowhere vanishing section. Intuitively, this means that all fibers are orientable in a compatible way. A family of real manifolds which underlies a family of complex manifolds is orientable, and carries a canonical orientation, exactly in the same way as the real manifold underlying a complex manifold is canonically oriented.

2.2. RELATIVE DE RHAM THEORY

Let $\pi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ be a family of real manifolds of relative dimension m; we denote by $\Omega^k_{X/Y}$ the kth exterior power of $\Omega^1_{X/Y}$ over $\mathcal{C}_X$, and call *relative k-forms* its sections. The exterior differential $d : \mathcal{C}_X \to \Omega^1_X$ induces, by composition with the projection $p : \Omega^1_X \to \Omega^1_{X/Y}$, a *relative differential* whose kernel is $\pi^{-1}\mathcal{C}_Y$ by definition. By extending in the usual way to the exterior algebra, one obtains a relative differential $d_r : \Omega^{k-1}_{X/Y} \to \Omega^k_{X/Y}$ for all $k > 0$, which makes the following diagram commute:

$$\begin{array}{ccc} \Omega^{k-1}_X & \xrightarrow{d} & \Omega^k_X \\ p\downarrow & & \downarrow p \\ \Omega^{k-1}_{X/Y} & \xrightarrow[d_r]{} & \Omega^k_{X/Y} \end{array} \quad . \tag{2.1}$$

On the other hand, the relative differential induces in each fiber a differential operator which coincides with the ordinary exterior differential.

We denote by $\mathcal{Z}^k_{X/Y}$ the sheaf of closed relative k-forms, i.e. the kernel of the morphism $d_r : \Omega^k_{X/Y} \to \Omega^{k+1}_{X/Y}$; in particular, $\mathcal{Z}^m_{X/Y} \equiv \Omega^m_{X/Y}$. A "relative Poincaré lemma" holds, that is, the following sequence of sheaves on X is exact [4,18]:

$$0 \longrightarrow \pi^{-1}\mathcal{C}_Y \longrightarrow \mathcal{C}_X \xrightarrow{d_r} \Omega^1_{X/Y} \xrightarrow{d_r} \cdots \xrightarrow{d_r} \Omega^m_{X/Y} \longrightarrow 0 \,. \tag{2.2}$$

So one has an acyclic resolution of the sheaf $\pi^{-1}\mathcal{C}_Y$.

Definition 2.1. The relative de Rham sheaf of degree k is the sheaf over Y

$$DR^k_{X/Y} \equiv \frac{\pi_* \mathcal{Z}^k_{X/Y}}{d_r \pi_* \Omega^{k-1}_{X/Y}} \,.$$

Equivalently, $DR^k_{X/Y}$ is the sheaf associated with the presheaf

$$V \rightsquigarrow \frac{\mathcal{Z}^k_{X/Y}(\pi^{-1}(V))}{d_r \Omega^{k-1}_{X/Y}(\pi^{-1}(V))} \,. \tag{2.3}$$

Thus, $DR^k_{X/Y}$ is the kth cohomology sheaf of the complex of sheaves over Y,

$$0 \longrightarrow \pi_*\pi^{-1}\mathcal{C}_Y \longrightarrow \pi_*\mathcal{C}_X \xrightarrow{d_r} \pi_*\Omega^1_{X/Y} \xrightarrow{d_r} \cdots \xrightarrow{d_r} \pi_*\Omega^m_{X/Y} \,.$$

If Y reduces to a point, $Y = \{y\}$, then $\Gamma(Y, DR^k_{X/Y})$ is the ordinary kth de Rham cohomology group of X.

Proposition 2.2. *For each $k \geq 0$ there is a canonical sheaf isomorphism*

$$DR^k_{X/Y} \xrightarrow{\sim} R^k\pi_*\pi^{-1}\mathcal{C}_Y\,.$$

Proof. Given a resolution $\mathcal{F}^\bullet$ of the sheaf $\pi^{-1}\mathcal{C}_Y$, there is a canonical morphism $\mathcal{H}^k(\pi_*\mathcal{F}^\bullet) \to R^k\pi_*\pi^{-1}\mathcal{C}_Y$ (abstract de Rham theorem); applying this to the resolution (2.2), one obtains a sheaf morphism $DR^k_{X/Y} \to R^k\pi_*\pi^{-1}\mathcal{C}_Y$. In order to prove that this is bijective, we can restrict to the stalks, thus getting

$$(DR^k_{X/Y})_y \simeq \frac{\Gamma(X_y, \mathcal{Z}^k_{X/Y})}{d_r\Gamma(X_y, \Omega^{k-1}_{X/Y})}, \qquad (R^k\pi_*\pi^{-1}\mathcal{C}_Y)_y \simeq H^k(X_y, \pi^{-1}\mathcal{C}_Y)\,.$$

The corresponding morphism between the right-hand sides is bijective by the de Rham theorem for sheaf cohomology, in that the sequence (2.2), when restricted to X_y, yields an acyclic resolution. □

Quite obviously, if $Y = \{y\}$, proposition 2.2 reduces to the ordinary de Rham theorem for X.

We now wish to investigate the relationship occurring between the relative de Rham cohomology of a family and the ordinary de Rham cohomology of the total space X.

Lemma 2.3. *There is a commutative diagram of* $\mathbb{C}$*-modules*

$$\begin{array}{ccccc} \Gamma(X, \mathcal{Z}^k_X) & \longrightarrow & H^k(X, \mathbb{C}) & \longrightarrow & \Gamma(Y, R^k\pi_*\mathbb{C}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ \Gamma(Y, \pi_*\mathcal{Z}^k_{X/Y}) & \longrightarrow & \Gamma(Y, DR^k_{X/Y}) & \xrightarrow{\sim} & \Gamma(Y, R^k\pi_*\pi^{-1}\mathcal{C}_Y) \end{array}\,.$$

Proof. Let us consider the square on the left; in view of the definition of relative differential, for any open $V \subset Y$ there is a commutative diagram

$$\begin{array}{ccccc} \Omega^{k-1}_X(\pi^{-1}(V)) & \xrightarrow{d} & \mathcal{Z}^k_X(\pi^{-1}(V)) & \longrightarrow & \dfrac{\mathcal{Z}^k_X(\pi^{-1}(V))}{d\,\Omega^{k-1}_X(\pi^{-1}(V))} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \Omega^{k-1}_{X/Y}(\pi^{-1}(V)) & \xrightarrow{d_r} & \mathcal{Z}^k_{X/Y}(\pi^{-1}(V)) & \longrightarrow & \dfrac{\mathcal{Z}^k_{X/Y}(\pi^{-1}(V))}{d_r\,\Omega^{k-1}_{X/Y}(\pi^{-1}(V))} \end{array}\,,$$

which yields a commutative diagram of presheaves. One concludes by passing to the associated sheaves in the bottom line and taking global sections.

The square on the right is induced by the diagram

$$\begin{array}{ccc} \pi_* \mathcal{Z}^k_X / d\, \pi_* \Omega^{k-1}_X & \longrightarrow & R^k \pi_* \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow \\ DR^k_{X/Y} & \longrightarrow & R^k \pi_* \pi^{-1} \mathcal{C}_Y \end{array} \quad ; \qquad (2.4)$$

here $\pi_* \mathcal{Z}^k_X / d\, \pi_* \Omega^{k-1}_X$ is the quotient *pre*sheaf, and the morphism in the first line is the canonical map to the associated sheaf. So, it is enough to prove that the diagram (2.4) is commutative, and this in turn can be proved by restricting to the stalks. One then considers the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \Omega^\bullet_X \\ & & \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & \pi^{-1}\mathcal{C}_Y & \longrightarrow & \Omega^\bullet_{X/Y} \end{array} ,$$

applies the direct image functor π_*, and takes cohomology, thus obtaining diagram (2.4) with the presheaf in the upper left corner replaced by the corresponding sheaf; the result we are looking for follows by composing with the canonical morphism mapping the presheaf to the sheaf. □

2.3. FIBERWISE INTEGRATION

Relative forms of top degree on a oriented family of real manifolds $\pi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ can be integrated "along the fibers" to yield a function on the base space Y. More precisely, if the family has relative dimension m, fiberwise integration is a sheaf morphism

$$\int_{X/Y} : \pi_* \Omega^m_{X/Y} \to \mathcal{C}_Y ,$$

defined as follows: if ω is a section of $\pi_* \Omega^m_{X/Y}$, for all y in the domain of ω one denotes by $\bar{\omega}_y \in \Gamma(X_y, \Omega^m_{X_y})$ the image of the germ $\omega_y \in \Gamma(X_y, \Omega^m_{X/Y})$; then

$$\Big(\int_{X/Y} \omega\Big)(y) = \int_{X_y} \bar{\omega}_y .$$

The integral on the right-hand side is the usual integral over a compact orientable manifold, and the fact that the left-hand side depends differentiably on y can be checked in local coordinates. If $Y = \{y\}$, fiberwise integration reduces to ordinary integration over X.

Proposition 2.4 (Stokes theorem). *For any section τ of $\pi_* \Omega^{m-1}_{X/Y}$ one has*

$$\int_{X/Y} d_r \tau = 0 .$$

Thus, fiberwise integration induces a morphism

$$\int_{X/Y} : DR^m_{X/Y} \to \mathcal{C}_Y \,. \tag{2.5}$$

Proof. In view of the definitions of fiberwise integration and relative differential, the first assertion reduces to Stokes theorem in each fiber. Then one has a morphism from the presheaf (2.3) into $\mathcal{C}_Y$, and one concludes by factorizing through the associated sheaf. □

Proposition 2.5. *The morphism (2.5) is bijective.*

Proof. The demonstration is the same as in the absolute case [16], for the functions on Y are constant as far as fiberwise integration is concerned. [#1] Indeed, the family being orientable, the morphism $\int_{X/Y}$ is not zero, i.e., there is a relative m-form ω such that $\int_{X/Y} \omega \neq 0$. One has to prove that the class $[\omega]$ is a generator of $DR^m_{X/Y}$, in the sense that for any other section ω' of $\pi_* \Omega^m_{X/Y}$ there are sections f of $\mathcal{C}_Y$ and τ of $\pi_* \Omega^{m-1}_{X/Y}$ such that $\omega' = f\omega + d_r \tau$. This fact is first proved locally on a coordinate cover (so that it reduces to the case $Y = \mathbb{R}^s$, $X = \mathbb{R}^s \times C$, where C is a compact orientable manifold) and then globalized by means of a partition of unity argument. □

Fiberwise integration can be used to introduce a *relative Poincaré duality*. One first defines a $\mathcal{C}_Y$-bilinear sheaf morphism

$$R^k \pi_* \pi^{-1} \mathcal{C}_Y \otimes_{\mathcal{C}_Y} R^{k'} \pi_* \pi^{-1} \mathcal{C}_Y \to R^{k+k'} \pi_* \pi^{-1} \mathcal{C}_Y \tag{2.6}$$

in the following way: $R^k \pi_* \pi^{-1} \mathcal{C}_Y$ is the sheaf associated with the presheaf $V \rightsquigarrow H^k(\pi^{-1}(V), \pi^{-1}\mathcal{C}_Y)$; a morphism like (2.6) is defined at the level of presheaves as the cup product, and then extended to the associated sheaves (by regarding the sheaves $R^k \pi_* \pi^{-1} \mathcal{C}_Y$ as $DR^k_{X/Y}$, this corresponds to the wedge product of relative forms). If $k' = m - k$, by composing the morphism (2.6) with fiberwise integration, one obtains a pairing

$$R^k \pi_* \pi^{-1} \mathcal{C}_Y \otimes_{\mathcal{C}_Y} R^{m-k} \pi_* \pi^{-1} \mathcal{C}_Y \to \mathcal{C}_Y \,, \tag{2.7}$$

which is non-degenerate, as one can show for instance by restricting to the stalks. In that case, the pairing (2.7) reduces to

$$H^k(X_y, \pi^{-1}\mathcal{C}_Y) \otimes_{(\mathcal{C}_Y)_y} H^{m-k}(X_y, \pi^{-1}\mathcal{C}_Y) \to (\mathcal{C}_Y)_y \,. \tag{2.8}$$

[#1] This is expressed by the equality $\int_{X/Y} (\pi^* f)\omega = f \int_{X/Y} \omega$.

Since $\pi^{-1}\mathcal{C}_Y|_{X_y}$ is the constant sheaf $(\mathcal{C}_Y)_y$, from the universal coefficient theorem one has $H^k(X_y, \pi^{-1}\mathcal{C}_Y) \simeq H^k(X_y, \mathbb{C}) \otimes_{\mathbb{C}} (\mathcal{C}_Y)_y$, so that (2.8) is the pairing

$$H^k(X_y, \mathbb{C}) \otimes_{\mathbb{C}} H^{m-k}(X_y, \mathbb{C}) \otimes_{\mathbb{C}} (\mathcal{C}_Y)_y \to (\mathcal{C}_Y)_y,$$

which is the $(\mathcal{C}_Y)_y$-bilinear extension of the Poincaré duality on the fiber X_y, and is therefore non-degenerate. □

2.4. RELATIVE DUALITY FOR ANALYTIC FAMILIES

Given a complex analytic family $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, one can introduce a *relative Serre duality*, which we would like to recall without providing any proof (cf. refs. [9,10,15]). Assume that the family has relative dimension n, denote by $\kappa_{X/Y}$ the sheaf of relative holomorphic n-forms, and let $\mathcal{M}$ and $\mathcal{N}$ be coherent $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules, respectively; then, denoting by R the operation of taking the derived functor, there is a canonical isomorphism of $\mathcal{O}_Y$-modules,

$$R\mathcal{H}om_{\mathcal{O}_Y}(R\pi_*\mathcal{M}, \mathcal{N}) \simeq R\pi_* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \kappa_{X/Y}(-n) \otimes_{\mathcal{O}_Y} \mathcal{N}).$$

One has then a convergent spectral sequence

$$\mathcal{E}_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_Y}(R^{n-q}\pi_*\mathcal{M}, \mathcal{N}) \Longrightarrow \mathcal{E}_\infty^{p+q} = \mathrm{Ext}^{p+q}_\pi(\mathcal{M}, \kappa_{X/Y} \otimes_{\mathcal{O}_Y} \mathcal{N}),$$

so that there is a natural morphism [#2]

$$\mathcal{E}_2^{0,q} = \mathcal{H}om_{\mathcal{O}_Y}(R^{n-q}\pi_*\mathcal{M}, \mathcal{N}) \to \mathcal{E}^q_\infty = \mathrm{Ext}^q_\pi(\mathcal{M}, \kappa_{X/Y} \otimes_{\mathcal{O}_Y} \mathcal{N}). \quad (2.9)$$

For $q = 0$, one deduces an isomorphism $\mathcal{H}om_{\mathcal{O}_Y}(R^n\pi_*\mathcal{M}, \mathcal{N}) \simeq \pi_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \kappa_{X/Y} \otimes_{\mathcal{O}_Y} \mathcal{N})$ for arbitrary $\mathcal{M}$ and $\mathcal{N}$, and one obtains in particular

$$\begin{aligned} R^n\pi_*\kappa_{X/Y} &\simeq (\pi_*\mathcal{O}_X)^\vee \\ &\simeq (\mathcal{O}_Y)^\vee \simeq \mathcal{O}_Y, \end{aligned} \quad (2.10)$$

where $^\vee$ denotes the dual module. The second isomorphism depends on π having universally connected fibers, while the first does not.

3. Relative theory of line bundles

3.1. PICARD SHEAF AND RELATIVE PICARD GROUP

The notion of relative Picard group, in the sense we are going to employ it, has been introduced by Grothendieck, both in the algebraic [7] and analytic case [8]. We would like to discuss that concept in the case of a family of complex manifolds, by making explicitly all the assumptions valid in that framework.

[#2] This morphism is clearly bijective for any $\mathcal{M}$ whenever $\mathcal{N}$ is a coherent injective sheaf, or for any $\mathcal{N}$ if all higher direct images $R^k\pi_*\mathcal{M}$ are locally free.

Our aim is to classify "families of line bundles" over the fibers of the morphism $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, where such "families" are, speaking heuristically, collections of line bundles, one on each fiber of π, with a certain compatibility condition. In order to do that, it is quite natural, on the analogy of what happens in the absolute case, to consider a "relative cohomology" of the sheaf $\mathcal{O}_X^*$ of invertible sections of $\mathcal{O}_X$. This relative cohomology is of course provided by the higher direct image functor $R^\bullet \pi_*$.

Definition 3.1. The Picard sheaf of the family $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is the sheaf $R^1 \pi_* \mathcal{O}_X^*$.

The group $\Gamma(Y, R^1\pi_*\mathcal{O}_X^*)$ is the *(restricted) relative Picard group* $\mathrm{Pic}(X/Y)$ (cf. Grothendieck [8]); if $Y = \{y\}$, the group $\mathrm{Pic}(X/Y)$ reduces to the ordinary Picard group $\mathrm{Pic}(X)$ of X. In the general case, the elements in the relative Picard group can be, according to the intuitive description reported above, interpreted as follows (this brief discussion is taken from ref. [8]). The sheaf $R^1\pi_*\mathcal{O}_X^*$ is associated with the presheaf $V \rightsquigarrow H^1(\pi^{-1}(V), \mathcal{O}_X^*) \simeq \mathrm{Pic}(\pi^{-1}(V))$ [group of isomorphism classes of line bundles over $\pi^{-1}(V)$]. The specification of an element of $\mathrm{Pic}(X/Y)$ is equivalent to the assignment of an open cover $\{V_i\}$ of Y and a collection of line bundles $\{\mathcal{L}_i\}$, each defined over $\pi^{-1}(V_i)$, such that for all pairs i, j, the bundles $\mathcal{L}_i|_{f^{-1}(V_i \cap V_j)}$ and $\mathcal{L}_j|_{f^{-1}(V_i \cap V_j)}$ are locally isomorphic relative to $V_i \cap V_j$, in the sense that any $y \in V_i \cap V_j$ has an open neighborhood $W \subset V_i \cap V_j$ such that $\mathcal{L}_i|_{f^{-1}(W)} \simeq \mathcal{L}_j|_{f^{-1}(W)}$.

The fact that $\mathcal{O}_Y \simeq \pi_*\mathcal{O}_X$, together with the Leray spectral sequence for the morphism π, gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\phi} \mathrm{Pic}(X/Y).$$

One denotes by $[\mathcal{L}]$ the image of an isomorphism class $\mathcal{L} \in \mathrm{Pic}(X)$ under ϕ. The morphism ϕ is in general not surjective, unless π admits a global section. Moreover, ϕ is functorial, in the following sense.

Proposition 3.2. *Given a morphism of complex analytic families $f : (X'/Y') \to (X/Y)$, there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow{\phi} & \mathrm{Pic}(X/Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Pic}(X') & \xrightarrow[\phi']{} & \mathrm{Pic}(X'/Y') \end{array} \quad . \tag{3.1}$$

Proof. It follows from the functoriality of the Leray sequence [11]. □

The morphism $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is actually induced by the inverse image of sheaves of $\mathcal{O}_X$-modules. In the particular case where the morphism is the immersion of a fiber, $i : (X_y/\{y\}) \hookrightarrow (X/Y)$, the diagram (3.1) and the identification $\mathrm{Pic}(X_y/\{y\}) \simeq \mathrm{Pic}(X_y)$ entails $[\mathcal{L}]_y \simeq \mathcal{L}_y$, where $[\mathcal{L}]_y$ is the stalk at y of the section $[\mathcal{L}] \in \mathrm{Pic}(X/Y)$, and $\mathcal{L}_y \in \mathrm{Pic}(X_y)$ is the line bundle $\mathcal{L}_y = \mathcal{L}|_{X_y} \otimes_{(\mathcal{O}_Y)_y} k(y) \simeq \mathcal{L}|_{X_y} \otimes_{\mathcal{O}_X|_{X_y}} \mathcal{O}_{X_y}$.

Definition 3.3. The relative Chern class $c_1(\lambda)$ of a section λ of the relative Picard sheaf $R^1\pi_*\mathcal{O}_X^*$ of the family $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is minus its image via the sheaf morphism

$$R^1\pi_*\mathcal{O}_X^* \to R^2\pi_*\mathbb{Z} \tag{3.2}$$

induced by the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp 2\pi i} \mathcal{O}_X^* \longrightarrow 0\,. \tag{3.3}$$

From (3.2) one obtains a morphism $c_1 : \Gamma(Y, R^1\pi_*\mathcal{O}_X^*) \to \Gamma(Y, R^2\pi_*\mathbb{Z})$, which we call *Chern class* as well; in the case $Y = \{y\}$, this morphism reduces to the ordinary Chern class $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$.

Proposition 3.4. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow{\phi} & \mathrm{Pic}(X/Y) \\ {\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle c_1} \\ H^2(X,\mathbb{Z}) & \longrightarrow & H^0(Y, R^2\pi_*\mathbb{Z}) \end{array} \quad . \tag{3.4}$$

Proof. It follows from the definition of Chern class. □

Proposition 3.5. *Given a morphism of complex analytic families $f : (X'/Y') \to (X/Y)$, there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(X/Y) & \xrightarrow{f^*} & \mathrm{Pic}(X'/Y') \\ {\scriptstyle c_1}\downarrow & & \downarrow{\scriptstyle c_1} \\ \Gamma(Y, R^2\pi_*\mathbb{Z}) & \xrightarrow[f^*]{} & \Gamma(Y', R^2\pi'_*\mathbb{Z}) \end{array} \quad . \tag{3.5}$$

Proof. One considers the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & f^{-1}\mathcal{O}_X & \xrightarrow{\exp 2\pi i} & f^{-1}\mathcal{O}_X^* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_{X'} & \xrightarrow{\exp 2\pi i} & \mathcal{O}_{X'}^* & \longrightarrow & 0 \end{array} \quad ;$$

by applying the higher direct image functor and taking global sections one concludes. □

From (3.5) applied in the case of the immersion of a fibre $i : (X_y/\{y\}) \hookrightarrow (X/Y)$, one has an identification $(c_1([\mathcal{L}]))_y \simeq c_1(\mathcal{L}_y)$ for each element $[\mathcal{L}] \in \mathrm{Pic}(X/Y)$.

If $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a family of Riemann surfaces, the Chern class $c_1 : R^1\pi_*\mathcal{O}_X^* \to R^2\pi_*\mathbb{Z}$ can be considered as a section in $R^2\pi_*\mathbb{C}$, due to the following result.

Proposition 3.6. *For a family of Riemann surfaces, the natural injection* $\mathbb{Z} \hookrightarrow \mathbb{C}$ *induces an immersion*

$$R^2\pi_*\mathbb{Z} \hookrightarrow R^2\pi_*\mathbb{C},$$

so that $\Gamma(Y, R^2\pi_*\mathbb{Z}) \hookrightarrow \Gamma(Y, R^2\pi_*\mathbb{C})$.

Proof. It is enough to prove that for any $y \in Y$ the morphism induced on the stalks $(R^2\pi_*\mathbb{Z})_y \to (R^2\pi_*\mathbb{C})_y$—that is, the morphism $H^2(X_y, \mathbb{Z}) \to H^2(X_y, \mathbb{C})$—is injective. This follows from the commutative diagram

$$\begin{array}{ccc} H^2(X_y,\mathbb{Z}) & \longrightarrow & H^2(X_y,\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{\text{inj}} & \mathbb{C} \end{array}, \tag{3.6}$$

where the vertical arrows are the Poincaré duality isomorphisms, which can be realized in $H^2(X_y, \mathbb{C})$ in terms of the integral over X_y. □

3.2. RELATIVE FLATNESS

We wish to generalize to the relative situation the well-known fact that a holomorphic line bundle on a Riemann surface is flat if and only if its Chern class vanishes. We recall that a line bundle over an n-dimensional analytic manifold $(X, \mathcal{O}_X)$ is said to be *flat* if its transition functions over a suitable trivializing open cover can be chosen so as to be locally constant. Equivalently, on can consider the immersion $\mathbb{C}^* \hookrightarrow \mathcal{O}_X^*$, and then (the isomorphism classes of) flat bundles are the elements in the image of the induced morphism $H^1(X, \mathbb{C}^*) \to H^1(X, \mathcal{O}_X^*)$. In the case of a family of analytic manifolds $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, the "relative constants" (in the sense that their relative differential vanishes) are the sections of $\pi^{-1}\mathcal{O}_Y$, which suggests the following definition.

Definition 3.7. A section of the relative Picard sheaf $R^1\pi_*\mathcal{O}_X^*$ of the family $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is said to be flat if it lies in the image of the morphism

$$R^1\pi_*\pi^{-1}\mathcal{O}_Y^* \to R^1\pi_*\mathcal{O}_X^* \tag{3.7}$$

induced by $\pi^{-1}\mathcal{O}_Y^* \hookrightarrow \mathcal{O}_X^*$.

Theorem 3.8. *Let $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a family of Riemann surfaces. Any $y \in Y$ has an open neighborhood V such that a section $\lambda \in \Gamma(V, R^1\pi_*\mathcal{O}_X^*)$ has vanishing relative Chern class if and only if it is flat.*

Proof. If $\kappa_{X/Y}$ denotes the sheaf of relative holomorphic one-forms, one has a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \kappa_{X/Y} & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi^{-1}\mathcal{O}_Y & \longrightarrow & \pi^{-1}\mathcal{O}_Y^* & \longrightarrow & 0
\end{array}$$

Applying the higher direct image functor, one obtains

$$\begin{array}{ccccccc}
R^2\pi_*\mathcal{O}_X & & & & & & \\
\uparrow & & & & & & \\
R^2\pi_*\pi^{-1}\mathcal{O}_Y & & & & & & \\
\alpha\uparrow & & & & & & \\
R^1\pi_*\kappa_{X/Y} & & & & & & \\
\uparrow & & & & & & \\
R^1\pi_*\mathcal{O}_X & \longrightarrow & R^1\pi_*\mathcal{O}_X^* & \xrightarrow{c_1} & R^2\pi_*\mathbb{Z} & \longrightarrow & R^2\pi_*\mathcal{O}_X \\
\uparrow & & \uparrow & & \| & & \uparrow \\
R^1\pi_*\pi^{-1}\mathcal{O}_Y & \longrightarrow & R^1\pi_*\pi^{-1}\mathcal{O}_Y^* & \longrightarrow & R^2\pi_*\mathbb{Z} & \xrightarrow{\beta} & R^2\pi_*\pi^{-1}\mathcal{O}_Y
\end{array} \tag{3.8}$$

The relative dimension of the family $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ being 1, one has $R^2\pi_*\mathcal{O}_X = 0$. If we prove that α is bijective and β is injective, diagram (3.8) yields

$$\begin{array}{ccccccc} 0 & & & & & & \\ \uparrow & & & & & & \\ R^1\pi_*\mathcal{O}_X & \longrightarrow & R^1\pi_*\mathcal{O}_X^* & \xrightarrow{c_1} & R^2\pi_*\mathbb{Z} & \longrightarrow & 0 \; ; \\ \uparrow & & \uparrow & & & & \\ R^1\pi_*\pi^{-1}\mathcal{O}_Y & \longrightarrow & R^1\pi_*\pi^{-1}\mathcal{O}_Y^* & \longrightarrow & 0 & & \end{array} \qquad (3.9)$$

the two lines of this diagram are exact, which implies our claim. Thus, the theorem reduces to the following two lemmas. □

Lemma 3.9. *The epimorphism $\alpha : R^1\pi_*\kappa_{X/Y} \to R^2\pi_*\pi^{-1}\mathcal{O}_Y$ is actually an isomorphism.*

Proof. From eq. (2.10) one has $R^1\pi_*\kappa_{X/Y} \simeq \mathcal{O}_Y$. Besides, $(R^2\pi_*\pi^{-1}\mathcal{O}_Y)_y \simeq H^2(X_y, \pi^{-1}\mathcal{O}_Y) \simeq H^2(X_y, (\mathcal{O}_Y)_y)$, since $\pi^{-1}\mathcal{O}_Y|_{X_y}$ is the constant sheaf $(\mathcal{O}_Y)_y$, and from the universal coefficient theorem one has $(R^2\pi_*\pi^{-1}\mathcal{O}_Y)_y \simeq H^2(X_y, \mathbb{C}) \otimes_{\mathbb{C}} (\mathcal{O}_Y)_y \simeq (\mathcal{O}_Y)_y$. Taking the quotient by the maximal ideal one gets an isomorphism $\alpha : \mathbb{C} \to \mathbb{C}$, and by Nakayama's lemma one concludes. □

One should remark that this result depends critically on π having universally connected fibers.

Lemma 3.10. *The morphism $\beta : R^1\pi_*\mathbb{Z} \to R^2\pi_*\pi^{-1}\mathcal{O}_Y$ is injective.*

Proof. It is enough to prove that for any $y \in Y$ the morphism induced on the stalk, $\beta : (R^2\pi_*\mathbb{Z})_y \to (R^2\pi_*\pi^{-1}\mathcal{O}_Y)_y$, is injective. A computation similar to that of the previous lemma enables us to write this morphism as

$$\beta : H^2(X_y, \mathbb{Z}) \to H^2(X_y, \mathbb{C}) \otimes_{\mathbb{C}} (\mathcal{O}_Y)_y ,$$

which can be identified with the cohomology morphism induced by $\mathbb{Z} \hookrightarrow (\mathcal{O}_Y)_y \simeq \mathbb{C} \otimes_{\mathbb{C}} (\mathcal{O}_Y)_y$, $m \mapsto m \otimes 1$. Then the claim follows from diagram (3.6). □

3.3. GAUSS–BONNET THEOREM FOR FAMILIES

Let $(X, \mathcal{O}_X)$ be a complex analytic manifold, and let $\mathcal{L}$ be a line bundle on it. It is a classical result that $c_1(\mathcal{L}) = (\mathrm{i}/2\pi)[K]$, where $[K]$ is the de Rham cohomology class of a (smooth) curvature form K on $\mathcal{L}$. If $(X, \mathcal{O}_X)$ is a compact

Riemann surface, the isomorphism $\int_X : H^2(X,\mathbb{C}) \xrightarrow{\sim} \mathbb{C}$ allows one to express this result in the form $c_1(\mathcal{L}) = (\mathrm{i}/2\pi)\int_X K$ (Gauss–Bonnet theorem). If $\pi : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a family of complex manifolds, and $\pi : (X,\mathcal{C}_X) \to (Y,\mathcal{C}_Y)$ is the underlying real family, the projection $p : \Omega^2_X \to \Omega^2_{X/Y}$ between the sheaves of smooth two-forms induces a morphism $p : H^2(X,\mathbb{C}) \to \Gamma(Y, DR^2_{X/Y})$, and one has an identification $p(c_1(\mathcal{L})) = (\mathrm{i}/2\pi)p([K])$. By applying lemma 2.3 and propositions 3.4 and 3.6, one deduces

$$c_1[\mathcal{L}] = (\mathrm{i}/2\pi)[p(K)] \in \Gamma(Y, DR^2_{X/Y}), \tag{3.10}$$

where $c_1[\mathcal{L}] \in \Gamma(Y, R^2\pi_*\mathbb{C})$ is regarded as a section in $\Gamma(Y, R^2\pi_*\pi^{-1}\mathcal{C}_Y) \simeq \Gamma(Y, DR^2_{X/Y})$. The relative two-form $p(K) \in \Gamma(Y, \pi_*\mathcal{Z}^2_{X/Y}) = \Gamma(X, \mathcal{Z}^2_{X/Y})$ can be interpreted as a "relative curvature" in the following way.

Definition 3.11. Given a family of complex manifolds $\pi : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ and a line bundle $\mathcal{L}$ over $(X,\mathcal{O}_X)$, let $\Omega^1_{X/Y}$ be the sheaf of relative differentials of the underlying real family; a (smooth) relative connection over $\mathcal{L}$ is a morphism of $\mathbb{C}$-modules

$$\nabla_r : \mathcal{L} \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{L} \tag{3.11}$$

which satisfies the Leibniz rule $\nabla_r(f\sigma) = d_r f \otimes \sigma + f\nabla_r(\sigma)$.

The projection of ordinary differentials onto relative differentials allows one to regard any relative connection as induced by an ordinary connection.

Lemma 3.12. *For any relative connection ∇_r on $\mathcal{L}$ there is an ordinary connection ∇ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\nabla} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L} \\ & {\scriptstyle\nabla_r}\searrow & \big\downarrow {\scriptstyle p\otimes \mathrm{Id}} \\ & & \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{L} \end{array} . \tag{3.12}$$

Proof. Fix a connection ∇^0 on $\mathcal{L}$, and let ∇^0_r be the corresponding relative connection, $\nabla^0_r = (p \otimes \mathrm{Id})\nabla^0$. For any relative connection ∇_r let α be a counterimage of $\nabla_r - \nabla^0_r$ under $p \otimes \mathrm{Id}$. Then $\nabla = \nabla^0 + \alpha$ is the desired connection. □

We extend ∇_r to a morphism $\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{L} \to \Omega^2_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{L}$ by letting $\nabla_r(\omega \otimes \sigma) = d_r\omega \otimes \sigma - \omega \wedge \nabla_r(\sigma)$.

Definition 3.13. The curvature K_r of a relative connection ∇_r over $\mathcal{L}$ is the morphism of $\mathbb{C}$-modules

$$K_r = \nabla^2_r : \mathcal{L} \to \Omega^2_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{L}. \tag{3.13}$$

By construction, if K is the curvature of the ordinary connection ∇ over $\mathcal{L}$ which induces ∇_r, one has $K_r = (p \otimes \mathrm{Id}) \circ K$. In particular, K_r is an $\mathcal{O}_X$-linear morphism, and therefore it determines a global section K_r of $\Omega^2_{X/Y}$. By eq. (2.1) K_r is also closed under the relative differential, and then $K_r \in \Gamma(Y, \pi_* \mathcal{Z}^2_{X/Y})$. The projection $[K_r] = [p(K)] = p([K]) \in \Gamma(Y, DR^2_{X/Y})$ does not depend on the relative connection over $\mathcal{L}$ (because $[K]$ does not), and from (3.10) one concludes that

$$c_1[\mathcal{L}] = (\mathrm{i}/2\pi)[K_r] \in \Gamma(Y, DR^2_{X/Y}). \tag{3.14}$$

If $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a family of Riemann surfaces, proposition 2.5 yields the identification $[K_r] = \int_{X/Y} K_r \in \Gamma(Y, \mathcal{C}_Y)$, which together with (3.14) proves a relative Gauss–Bonnet theorem.

Theorem 3.14. *Let $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a family of Riemann surfaces, $\mathcal{L}$ a line bundle over $(X, \mathcal{O}_X)$, let $[\mathcal{L}]$ be its image in* Pic(X/Y), *and K_r the curvature of any relative connection over $\mathcal{L}$; then*

$$c_1([\mathcal{L}]) = (\mathrm{i}/2\pi) \int_{X/Y} K_r. \qquad \square$$

Corollary 3.15. *Assume that ϕ :* Pic $X \to$ Pic(X/Y) *is surjective, and let $\lambda \in$* Pic(X/Y). *Then $c_1(\lambda) = (\mathrm{i}/2\pi) \int_{X/Y} K_r$, where K_r is the curvature of a relative connection on any line bundle on $(X, \mathcal{O}_X)$ whose image in* Pic(X/Y) *is λ.* $\square$

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